RAMANUJAN AND CRANKS

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Abstract The existence of the crank was first conjectured by F. J. Dyson in 1944 and was later established by G. E. Andrews and F. G. Garvan in 1987. However, much earlier, in his lost notebook, Ramanujan studied the generating function $F_a(q)$ for the crank and offered several elegant claims about it, although it seems unlikely that he was familiar with all the combinatorial implications of the crank. In particular, Ramanujan found several congruences for $F_a(q)$ in the ring of formal power series in the two variables a and q. An obscure identity found on page 59 of the lost notebook leads to uniform proofs of these congruences. He also studied divisibility properties for the coefficients of $F_a(q)$ as a power series in q. In particular, he provided ten lists of coefficients which he evidently thought exhausted these divisibility properties. None of the conjectures implied by Ramanujan's tables have been proved.

1. Introduction

In attempting to find a combinatorial interpretation for Ramanujan's famous congruences for the partition function $p(n)$, the number of ways of representing the positive integer n as a sum of positive integers, in 1944, F. J. Dyson [7] defined the *rank of a partition* to be the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m, and let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Then Dyson conjectured that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \qquad 0 \leq k \leq 4, \tag{1.1}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \qquad 0 \leq k \leq 6, \tag{1.2}$$

which yield combinatorial interpretations of Ramanujan's famous congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, respectively. These conjectures, as well as further conjectures of Dyson, were first proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [4] in 1954. The generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m,n) a^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq;q)_n (q/a;q)_n}, \tag{1.3}$$

where $|q| < 1$ and $|q| < |a| < 1/|q|$. Although, to the best of our knowledge, Ramanujan was unaware of the concept of the rank of a partition, he recorded theorems on its generating function in his lost notebook; in particular, see [20, p. 20].

The corresponding analogue does not hold for $p(11n+6) \equiv 0 \pmod{11}$, and so Dyson conjectured the existence of a *crank*. In his doctoral dissertation [11], F. G. Garvan defined vector partitions which became the forerunners of the crank. The *true crank* was discovered by G. E. Andrews and Garvan on June 6, 1987, at a student dormitory at the University of Illinois.

Definition 1.1. *For a partition π, let $\lambda(n)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be*

$$c(\pi) = \begin{cases} \lambda(\pi), & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0. \end{cases} \tag{1.4}$$

For $n \neq 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n = 1$, we set

$$M(0,1) = -1, M(-1,1) = M(1,1) = 1, \text{and } M(m,1) = 0 \text{ otherwise.}$$

Let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t. The main theorem of Andrews and Garvan [2] relates $M(m,n)$ with vector partitions. In particular, the generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n)a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{1.5}$$

The crank not only leads to a combinatorial interpretation of $p(11n+6) \equiv 0 \pmod{11}$, as predicted by Dyson, but also to similar interpretations for $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

Theorem 1.2. *With $M(m,t,n)$ defined above,*

$$\begin{aligned} M(k,5,5n+4) &= \frac{p(5n+4)}{5}, & 0 \le k \le 4, \\ M(k,7,7n+5) &= \frac{p(7n+5)}{7}, & 0 \le k \le 6, \\ M(k,11,11n+6) &= \frac{p(11n+6)}{11}, & 0 \le k \le 10. \end{aligned}$$

An excellent introduction to cranks can be found in Garvan's survey paper [12]. Also, see [3] for an interesting article on relations between the ranks and cranks of partitions.

2. Entries on Pages 179 and 180

At the top of page 179 in his lost notebook [20], Ramanujan defines a function $F(q)$ and coefficients λ_n, $n \geq 0$, by

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n. \tag{2.1}$$

Thus, $F_a(q)$ is the generating function for cranks, and by (1.5), for $n > 1$,

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n)a^m.$$

He then offers two congruences for $F_a(q)$. These congruences, like others in the sequel, are to be regarded as congruences in the ring of formal power series in the two variables a and q. First, however, we need to define Ramanujan's theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1, \tag{2.2}$$

which satisfies the Jacobi triple product identity [5, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.3}$$

The two congruences are then given by the following two theorems.

Theorem 2.1.

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3,-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right)\sqrt{q}\,\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \pmod{a^2 + \frac{1}{a^2}}. \tag{2.4}$$

Theorem 2.2.

$$\begin{aligned} F_a(q^{1/3}) \equiv\ & \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\ & + \left(a - 1 + \frac{1}{a}\right) q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\ & + \left(a^2 + \frac{1}{a^2}\right) q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \pmod{a^3 + 1 + \frac{1}{a^3}}. \end{aligned} \tag{2.5}$$

Note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \pmod{\lambda_2}$ and $a^8 \equiv 1 \pmod{\lambda_2}$. Thus, in (2.4), a behaves like a primitive 8th root of unity modulo λ_2. On the other hand, $\lambda_3 = a^3 + 1 + a^{-3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1 \pmod{\lambda_3}$. So in (2.5), a behaves like a primitive 9th root of unity modulo λ_3.

This now leads us to the following definition.

Definition 2.3. *Let $P(q)$ denote any power series in q. Then the t-dissection of P is given by*

$$P(q) =: \sum_{k=0}^{t-1} q^k P_k(q^t). \tag{2.6}$$

Thus, if we let $a = \exp(2\pi i/8)$ and replace q by q^2, (2.4) implies the 2-dissection of $F_a(q)$, while if we let $a = \exp(2\pi i/9)$ and replace q by q^3, (2.5) implies the 3-dissection of $F_a(q)$. The first proofs of (2.4) and (2.5) in the forms where a is replaced by the respective primitive root of unity were given by Garvan [14]; his proof of (2.5) uses a Macdonald identity for the root system A_2.

3. Entries on Pages 18 and 20

Ramanujan gives the 5-dissection of $F_a(q)$ on pages 18 and 20 of his lost notebook [20], with the better formulation on page 20. It is interesting that Ramanujan does not give the two variable form, analogous to those in (2.4) and (2.5), from which the 5-dissection would follow by setting a to be a primitive fifth root of unity. Proofs of the 5-dissection have been given by Garvan [13] and A. B. Ekin [9]. To describe this dissection, we first set

$$f(-q) := f(-q, -q^2) = (q;q)_\infty, \tag{3.1}$$

by (2.3).

Theorem 3.1. *If ζ is a primitive fifth root of unity and $f(-q)$ is defined by* (3.1), *then*

$$\begin{aligned} F_\zeta(q) = & \frac{f(-q^{10}, -q^{15})}{f^2(-q^5, -q^{20})} f^2(-q^{25}) \\ & + (\zeta - 1 + \zeta^{-1}) q \frac{f^2(-q^{25})}{f(-q^5, -q^{20})} \\ & + (\zeta^2 + \zeta^{-2}) q^2 \frac{f^2(-q^{25})}{f(-q^{10}, -q^{15})} \\ & + (\zeta^3 + 1 + \zeta^{-3}) q^3 \frac{f(-q^5, -q^{20})}{f^2(-q^{10}, -q^{15})} f^2(-q^{25}). \end{aligned}$$

For completeness, we state Theorem 3.1 in the two variable form as a congruence. But first, for brevity, it will be convenient to define

$$A_n := a^n + a^{-n} \tag{3.2}$$

and

$$S_n := \sum_{k=-n}^{n} a^k. \tag{3.3}$$

Theorem 3.2. *With $f(-q)$, A_n, and S_n defined by* (3.1)–(3.3), *respectively,*

$$\begin{aligned}
F_a(q) \equiv & \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})} f^2(-q^{25}) \\
& + (A_1-1)q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})} + A_2 q^2 \frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} \\
& + (A_3+1)q^3 \frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})} f^2(-q^{25}) \pmod{S_2}.
\end{aligned}$$

As we have seen, by letting a be a root of unity, we can derive a dissection from a congruence in the ring of formal power series in two variables. In fact, the converse is true, and this is proved in [6].

4. Entries on Pages 70 and 71

The first explicit statement and proof of the 7-dissection of $F_a(q)$ was given by Garvan [13, Thm. 5.1]; another proof was later found by Ekin [9]. Although Ramanujan did not state the 7-dissection of $F_a(q)$, he clearly knew it, because the six quotients of theta functions that appear in the 7-dissection are found on the bottom of page 71 (written upside down) in his lost notebook. We record the two variable form here.

Theorem 4.1. *With $f(a,b)$ defined by* (2.2), *$f(-q)$ defined by* (3.1), *and A_n and S_n defined by* (3.2) *and* (3.3), *respectively,*

$$\begin{aligned}
F_a(q) \equiv & \frac{1}{f(-q^7)} \{ f^2(-q^{21},-q^{28}) + (A_1-1)qf(-q^{14},-q^{35})f(-q^{21},-q^{28}) \\
& + A_2q^2f^2(-q^{14},-q^{35}) + (A_3+1)q^3f(-q^7,-q^{42})f(-q^{21},-q^{28}) \\
& - A_1q^4f(-q^7,-q^{42})f(-q^{14},-q^{35}) - (A_2+1)q^6f^2(-q^7,-q^{42})\} \\
& \pmod{S_3}.
\end{aligned}$$

The first appearance of the 11-dissection of $F_a(q)$ in the literature also can be found in Garvan's paper [13, Thm. 6.7]. However, again, it is very likely that Ramanujan knew the 11-dissection, since he offers the quotients of theta functions which appear in the 11-dissection on page 70 of his lost notebook [20]. Further proofs were found by Ekin [8], [9], and a reformulation of Garvan's result was given by M. D. Hirschhorn [15]. We state the 11-dissection in the two variable form as a congruence.

Theorem 4.2. *With A_n and S_n defined by* (3.2) *and* (3.3), *respectively,*

$$\begin{aligned}F_a(q) \equiv \frac{1}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}\Big\{&ABCD + (A_1-1)\,qA^2BE\\&+ A_2q^2AC^2D + (A_3+1)\,q^3ABD^2\\&+ (A_2+A_4+1)\,q^4ABCE - (A_2+A_4)\,q^5B^2CE\\&+ (A_1+A_4)\,q^7ABDE - (A_2+A_5+1)\,q^{19}CDE^2\\&- (A_4+1)\,q^9ACDE - A_3q^{10}BCDE\Big\} \pmod{S_5},\end{aligned}$$

where $A = f(-q^{55}, -q^{66})$, $B = f(-q^{77}, -q^{44})$, $C = f(-q^{88}, -q^{33})$, $D = f(-q^{99}, -q^{22})$, *and* $E = f(-q^{110}, -q^{11})$.

The present authors have recently given two proofs of each of Theorems 2.1, 2.2, 3.2, 4.1, and 4.2 in [6]. Our first proofs of each theorem use a method of "rationalization" which is like the method employed by Garvan [13], [14] in proving the dissections where a is replaced with a primitive root of unity. Our second method employs a formula found on page 59 in Ramanujan's lost notebook [20]. In fact, as we shall see in the next section, Ramanujan actually does not record a formula, but instead records "each side" without stating an equality.

5. Entries on Pages 58 and 59

On page 58 in his lost notebook [20], Ramanujan recorded the following power series:

$$\begin{aligned}&1 + q(a_1-1) + q^2a_2 + q^3(a_3+1) + q^4(a_4+a_2+1)\\&+ q^5(a_5+a_3+a_1+1) + q^6(a_6+a_4+a_3+a_2+a_1+1)\\&+ q^7(a_3+1)(a_4+a_2+1) + q^8a_2(a_6+a_4+a_3+a_2+a_1+1)\\&+ q^9a_2(a_3+1)(a_4+a_2+1) + q^{10}a_2(a_3+1)(a_5+a_3+a_1+1)\\&+ q^{11}a_1a_2(a_8+a_5+a_4+a_3+a_2+a_1+2)\\&+ q^{12}(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\&\quad\times(a_4-2a_3+2a_2-a_1+1)\\&+ q^{13}(a_1-1)(a_2-a_1+1)\\&\quad\times(a_{10}+2a_9+2a_8+2a_7+2a_6+4a_5+6a_4+8a_3+9a_2+9a_1+9)\\&+ q^{14}(a_2+1)(a_3+1)(a_4+a_2+1)(a_5-a_3+a_1+1)\\&+ q^{15}a_1a_2(a_5+a_4+a_3+a_2+a_1+1)(a_7-a_6+a_4+a_1)\\&+ q^{16}(a_3+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\&\quad\times(a_5-2a_4+2a_3-2a_2+3a_1-3)\end{aligned}$$

$$
\begin{aligned}
&+q^{17}(a_2+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1)(a_7-a_6+a_3+a_1-1)\\
&+q^{18}(a_4+a_2+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&\quad\times(a_6-2a_5+a_4+a_3-a_2+1)\\
&+q^{19}a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1)\\
&\quad\times(a_9-a_7+a_4+2a_3+a_2-1)\\
&+q^{20}(a_2-a_1+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&\quad\times(a_{10}+a_6+a_4+a_3+2a_2+2a_1+3)\\
&+q^{21}a_1a_2(a_3+1)(a_2-a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&\quad\times(a_8-a_6+a_4+a_1+2)\\
&+\cdots. \qquad (5.1)
\end{aligned}
$$

Although Ramanujan did not indicate the meaning of his notation a_n, in fact,

$$a_n := a^n + a^{-n}, \qquad (5.2)$$

and indeed Ramanujan has written out the first 21 coefficients in the power series representation of the crank $F_a(q)$. (We have corrected a misprint in the coefficient of q^{21}.)

On the following page, beginning with the coefficient of q^{13}, Ramanujan listed some (but not necessarily all) of the factors of the coefficients up to q^{26}. The factors he recorded are

$$
\begin{aligned}
&13.\quad (a_1-1)(a_2-a_1+1) \qquad (5.3)\\
&14.\quad (a_2+1)(a_3+1)(a_4+a_2+1)\\
&15.\quad a_1a_2(a_5+a_4+a_3+a_2+a_1+1)\\
&16.\quad (a_3+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&17.\quad (a_2+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&18.\quad (a_4+a_2+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&19.\quad a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1)\\
&20.\quad (a_3+1)(a_2-a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&21.\quad a_1a_2(a_3+1)(a_2-a_1+1)(a_5+a_4+a_3+a_2+a_1+1)\\
&22.\quad a_2(a_3+1)(a_1-1)\\
&23.\quad (a_1-1)(a_4+a_2+1)\\
&24.\quad (a_3+1)(a_4+a_2+1)(a_3+a_2+a_1+1)\\
&25.\quad a_2(a_1-1)(a_5+a_4+a_3+a_2+a_1+1)\\
&26.\quad a_2(a_3+1)(a_3+a_2+a_1+1).
\end{aligned}
$$

Ramanujan did not indicate why he recorded only these factors. However, it can be noted that in each case he recorded linear factors only when the leading index is ≤ 5. To the left of each n, $15 \leq n \leq 26$, are the unexplained numbers 16×16, undecipherable, $27 \times 27, -25, 49, -7 \cdot 19, 9, -7, -9, -11 \cdot 15, -11$, and -4, respectively.

6. Congruences for the Coefficients λ_n on Pages 179 and 180

On pages 179 and 180 in his lost notebook [20], Ramanujan offers ten tables of indices of coefficients λ_n satisfying certain congruences. On page 61 in [20], he offers rougher drafts of nine of the ten tables; Table 6 is missing on page 61. Unlike the tables on pages 179 and 180, no explanations are given on page 61. Clearly, Ramanujan calculated factors well beyond the factors recorded on pages 58 and 59 of his lost notebook given in Section 5.5. To verify Ramanujan's claims, we calculated λ_n up to $n = 500$ with the use of Maple V. Ramanujan evidently thought that each table is complete in that there are no further values of n for which the prescribed divisibility property holds. However, we are unable to prove any of these assertions.

Table 1. $\lambda_n \equiv 0 \pmod{a^2 + \frac{1}{a^2}}$

Thus, Ramanujan indicates which coefficients λ_n have a_2 as a factor. The 47 values of n with a_2 as a factor of λ_n are

$$2, 8, 9, 10, 11, 15, 19, 21, 22, 25, 26, 27, 28, 30, 31, 34, 40, 42, 45,$$
$$46, 47, 50, 55, 57, 58, 59, 62, 66, 70, 74, 75, 78, 79, 86, 94, 98,$$
$$106, 110, 122, 126, 130, 142, 154, 158, 170, 174, 206.$$

Replacing q by q^2 in (2.4), we see that Table 1 contains the degree of q for those terms with zero coefficients for both

$$\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty} \qquad \text{and} \qquad q\frac{f(-q^2, -q^{14})}{(-q^4; q^4)_\infty}. \tag{6.1}$$

Table 2. $\lambda_n \equiv 1 \pmod{a^2 + \frac{1}{a^2}}$

To interpret this table properly, we return to the congruence given in (2.4). Replacing q by q^2, we see that Ramanujan has recorded all the degrees of q of the terms (except for the constant term) with coefficients equal to 1 in the power series expansion of

$$\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty}. \tag{6.2}$$

The 27 values of n given by Ramanujan are

$$14, 16, 18, 24, 32, 48, 56, 72, 82, 88, 90, 104, 114, 138, 146,$$
$$162, 178, 186, 194, 202, 210, 218, 226, 234, 242, 250, 266.$$

Table 3. $\lambda_n \equiv -1 \pmod{a^2 + \frac{1}{a^2}}$

This table is to be understood in the same way as the previous table, except that now Ramanujan is recording the indices of those terms with coefficients equal to -1 in the power series expansion of (6.2). Here Ramanujan missed one value, namely, $n = 214$. The 27 (not 26) values of n are then given by

$$4, 6, 12, 20, 36, 38, 44, 52, 54, 60, 68, 76, 92, 102, 118,$$
$$134, 150, 166, 182, 190, 214, 222, 238, 254, 270, 286, 302.$$

Table 4. $\lambda_n \equiv a - 1 + \frac{1}{a} \pmod{a^2 + \frac{1}{a^2}}$

We again return to the congruence given in (2.4). Note that $a-1+1/a$ occurs as a factor of the second expression on the right side. Thus, replacing q by q^2, Ramanujan records the indices of all terms of

$$q\frac{f(-q^2, -q^{14})}{(-q^4; q^4)_\infty} \tag{6.3}$$

with coefficients that are equal to 1. The 22 values of n which give the coefficient 1 are equal to

$$1, 7, 17, 23, 33, 39, 41, 49, 63, 71, 73, 81,$$
$$87, 89, 95, 105, 111, 119, 121, 127, 143, 159.$$

Table 5. $\lambda_n \equiv -\left(a - 1 + \frac{1}{a}\right) \pmod{a^2 + \frac{1}{a^2}}$

The interpretation of this table is analogous to the preceding one. Now Ramanujan determines those coefficients in the expansion of (6.3) which are equal to -1. His table of 23 values of n includes

$$3, 5, 13, 29, 35, 37, 43, 51, 53, 61, 67, 69, 77,$$
$$83, 85, 91, 93, 99, 107, 115, 123, 139, 155.$$

Table 6. $\lambda_n \equiv 0 \pmod{a + \frac{1}{a}}$

Ramanujan thus gives here those coefficients which have a_1 as a factor. There are only three values, namely, when n equals

$$11, 15, 21.$$

These three values can be discerned from the table on page 59 of the lost notebook.

From the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)_\infty}{(-q^2;q^2)_\infty} = \frac{f(-q)f(-q^2)}{f(-q^4)} \pmod{a+\frac{1}{a}},$$

where $f(-q)$ is defined by (3.1), we see that in Table 6 Ramanujan recorded the degree of q for the terms with zero coefficients in the power series expansion of

$$\frac{f(-q)f(-q^2)}{f(-q^4)}. \tag{6.4}$$

For the next three tables, it is clear from the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q^2;q^2)_\infty}{(-q^3;q^3)_\infty} = \frac{f(-q^2)f(-q^3)}{f(-q^6)} \pmod{a-1+\frac{1}{a}},$$

that Ramanujan recorded the degree of q for the terms with coefficients 0, 1, and -1, respectively, in the power series expansion of

$$\frac{f(-q^2)f(-q^3)}{f(-q^6)}. \tag{6.5}$$

Table 7. $\lambda_n \equiv 0 \pmod{a-1+\frac{1}{a}}$

The 19 values satisfying the congruence above are, according to Ramanujan,

$$1, 6, 8, 13, 14, 17, 19, 22, 23, 25,$$
$$33, 34, 37, 44, 46, 55, 58, 61, 82.$$

Table 8. $\lambda_n \equiv 1 \pmod{a-1+\frac{1}{a}}$

The 26 values of n found by Ramanujan are

$$5, 7, 10, 11, 12, 18, 24, 29, 30, 31, 35, 41, 42, 43,$$
$$47, 49, 53, 54, 59, 67, 71, 73, 85, 91, 97, 109.$$

As in Table 2, Ramanujan ignored the value $n = 0$.

Table 9. $\lambda_n \equiv -1 \pmod{a-1+\frac{1}{a}}$

The 26 values of n found by Ramanujan are

$$2, 3, 4, 9, 15, 16, 20, 21, 26, 27, 28, 32, 38, 39,$$
$$40, 52, 56, 62, 64, 68, 70, 76, 94, 106, 118, 130.$$

Table 10. $\lambda_n \equiv 0 \pmod{a+1+\frac{1}{a}}$

Ramanujan has but two values of n such that λ_n satisfies the congruence above, and they are when n equals

$$14, 17.$$

From the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty} = \frac{f^2(-q)}{f(-q^3)} \pmod{a+1+\frac{1}{a}},$$

it is clear that Ramanujan recorded the degree of q for the terms with zero coefficients in the power series expansion of

$$\frac{f^2(-q)}{f(-q^3)}. \tag{6.6}$$

The infinite products in (6.2)–(6.6) do not appear to have monotonic coefficients for sufficiently large n. However, if these infinite products are dissected properly, then we conjecture that the coefficients in the dissections are indeed monotonic. Hence, for (6.2), (6.3), (6.4), (6.5), and (6.6), we must study, respectively, the dissections of

$$\frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty}, \qquad \frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty},$$

$$\frac{f(-q)f(-q^2)}{f(-q^4)}, \qquad \frac{f(-q^2)f(-q^3)}{f(-q^6)}, \qquad \frac{f^2(-q)}{f(-q^3)}.$$

For each of the five products given above, we have determined certain dissections.

We require an addition theorem for theta functions found in Chapter 16 of Ramanujan's second notebook [19], [5, p. 48, Entry 31]. Our applications of this lemma lead to the desired dissections.

Lemma 6.1. *If $U_n = \alpha^{n(n+1)/2}\beta^{n(n-1)/2}$ and $V_n = \alpha^{n(n-1)/2}\beta^{n(n+1)/2}$ for each integer n, then*

$$f(U_1, V_1) = \sum_{k=0}^{N-1} U_k f\left(\frac{U_{N+k}}{U_k}, \frac{V_{N-k}}{U_k}\right). \tag{6.7}$$

Setting $(\alpha, \beta, N) = (-q^6, -q^{10}, 4)$ and $(-q^4, -q^{12}, 2)$ in (6.7), we obtain, respectively,

$$f(-q^6, -q^{10}) = A - q^6 B - q^{10} C + q^{28} D, \tag{6.8}$$

$$f(-q^4,-q^{12}) = f(q^{24},q^{40}) - q^4 f(q^8,q^{56}), \tag{6.9}$$

where $A := f(q^{120},q^{136})$, $B := f(q^{72},q^{184})$, $C := f(q^{56},q^{200})$, and $D := f(q^8,q^{248})$.

Setting $(\alpha,\beta,N) = (-q,-q^2,3)$ in (6.7), we obtain

$$f(-q) = f(-q^{12},-q^{15}) - qf(-q^6,-q^{21}) - q^2 f(-q^3,-q^{24}). \tag{6.10}$$

For (6.2), the 8-dissection (with, of course, the odd powers missing) is given by

$$\begin{aligned}
\frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty} &= \frac{f(-q^6,-q^{10})f(-q^4,-q^{12})}{f(-q^{16})} \\
&= \frac{1}{f(-q^{16})}\left\{A - q^6B - q^{10}C + q^{28}D\right\} \\
&\quad\times \left\{f(q^{24},q^{40}) - q^4 f(q^8,q^{56})\right\} \\
&= \frac{1}{f(-q^{16})}\Big\{Af(q^{24},q^{40}) - q^{32}Df(q^8,q^{56}) \\
&\quad + q^2\left[q^8Bf(q^8,q^{56}) - q^8Cf(q^{24},q^{40})\right] \\
&\quad + q^4\left[-Af(q^{24},q^{40}) + q^{24}Df(q^8,q^{56})\right] \\
&\quad + q^6\left[-Bf(q^{24},q^{40}) + q^8Cf(q^8,q^{56})\right]\Big\},
\end{aligned}$$

where we have applied (6.8) and (6.9) in the penultimate equality.

For (6.6), we have the 3-dissection,

$$\begin{aligned}
\frac{f^2(-q)}{f(-q^3)} &= \frac{1}{(q^3;q^3)_\infty}\left\{f(-q^{12},-q^{15}) - qf(-q^6,-q^{21}) - q^2f(-q^3,-q^{24})\right\}^2 \\
&= \frac{1}{(q^3;q^3)_\infty}\Big\{f^2(-q^{12},-q^{15}) + 2q^3 f(-q^6,-q^{21})f(-q^3,-q^{24}) \\
&\quad - q\left[2f(-q^{12},-q^{15})f(-q^6,-q^{21}) - q^3f^2(-q^3,-q^{24})\right] \\
&\quad + q^2\left[f^2(-q^6,-q^{21}) - 2f(-q^{12},-q^{15})f(-q^3,-q^{24})\right]\Big\},
\end{aligned}$$

where we have applied (6.10) in the first equality. For (6.3), (6.4), and (6.5), we have derived an 8-dissection, a 4-dissection, and a 6-dissection, respectively. Furthermore, we make the following conjecture.

Conjecture 6.2. *Each component of each of the dissections for the five products given above has monotonic coefficients for powers of q above* 1400.

We have checked the coefficients for each of the five products up to $n = 2000$. For each product, we give below the values of n after which their dissections appear to be monotonic and strictly monotonic, respectively.

(6.2)	1262	1374
(6.3)	719	759
(6.4)	149	169
(6.5)	550	580
(6.6)	95	95

Our conjectures on the dissections of (6.4), (6.5), and (6.6) have motivated the following stronger conjecture.

Conjecture 6.3. *For any positive integers α and β, each component of the $(\alpha+\beta+1)$-dissection of the product*

$$\frac{f(-q^{\alpha})f(-q^{\beta})}{f(-q^{\alpha+\beta+1})}$$

has monotonic coefficients for sufficiently large powers of q.

We remark that our conjectures for (6.4), (6.5), and (6.6) are then the special cases of Conjecture 6.3 when we set $(\alpha,\beta) = (1,2),(2,3)$, and $(1,1)$, respectively.

Setting $(\alpha,\beta,N) = (-q^6,-q^{10},2)$ and $(-q^2,-q^{14},2)$ in (6.7), we obtain, respectively,

$$f(-q^6,-q^{10}) = f(q^{28},q^{36}) - q^6 f(q^4,q^{60}) \tag{6.11}$$

and

$$f(-q^2,-q^{14}) = f(q^{20},q^{44}) - q^2 f(q^{12},q^{52}). \tag{6.12}$$

After reading our conjectures for (6.2) and (6.3), Garvan made the following stronger conjecture.

Conjecture 6.4. *Define b_n by*

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^n &= \frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty} + q\frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty}\\ &= \frac{f(q^{28},q^{36})}{(-q^4;q^4)_\infty} + q\frac{f(q^{20},q^{44})}{(-q^4;q^4)_\infty} - q^6\frac{f(q^4,q^{60})}{(-q^4;q^4)_\infty}\\ &\quad - q^3\frac{f(q^{12},q^{52})}{(-q^4;q^4)_\infty},\end{aligned}$$

where we have applied (6.11) *and* (6.12) *in the last equality. Then*

$$\begin{aligned}
(-1)^n b_{4n} &\geq 0, &\quad &\textit{for all } n \geq 0,\\
(-1)^n b_{4n+1} &\geq 0, &\quad &\textit{for all } n \geq 0,\\
(-1)^n b_{4n+2} &\geq 0, &\quad &\textit{for all } n \geq 0, n \neq 3,\\
(-1)^{n+1} b_{4n+3} &\geq 0, &\quad &\textit{for all } n \geq 0.
\end{aligned}$$

Furthermore, each of these subsequences are eventually monotonic.

It is clear that the monotonicity of the subsequences in Conjecture 6.4 implies the monotonicity of the dissections of (6.2) and (6.3) as stated in Conjecture 6.2.

In [1], Andrews and R. Lewis made three conjectures on the inequalities between the rank counts $N(m, t, n)$ and between the crank counts $M(m, t, n)$. Two of them, [1, Conj. 2 and Conj. 3] directly imply that Tables 10 and 6, respectively, are complete. Recently, using the circle method, D. M. Kane [16] proved the former conjecture. It follows immediately from [16, Cor. 2] that Table 10 is complete.

7. Page 182: Partitions and Factorizations of Crank Coefficients

On page 182 in his lost notebook [20], Ramanujan returns to the coefficients λ_n in the generating function (2.1) of the crank. He factors λ_n, $1 \leq n \leq 21$, as before, but singles out nine particular factors by giving them special notation. The criterion that Ramanujan apparently uses is that of multiple occurrence, i.e., each of these nine factors appears more than once in the 21 factorizations, while other factors not favorably designated appear only once. Ramanujan uses these factorizations to compute $p(n)$, which, of course, arises from the special case $a = 1$ in (2.1), i.e.,

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n, \qquad |q| < 1.$$

Ramanujan evidently was searching for some general principles or theorems on the factorization of λ_n so that he could not only compute $p(n)$ but say something about the divisibility of $p(n)$. No theorems are stated by Ramanujan. Is it possible to determine that certain factors appear in some precisely described infinite family of values of λ_n? It would be interesting to speculate on the motivations which led Ramanujan to make these factorizations.

The factors designated by Ramanujan are

$$\rho_1 = a_1 - 1,$$

$$\begin{aligned}
\rho &= a_2 - a_1 + 1, \\
\rho_2 &= a_2, \\
\rho_3 &= a_3 + 1, \\
\rho_4 &= a_1 a_2, \\
\rho_5 &= a_4 + a_2 + 1, \\
\rho_7 &= a_3 + a_2 + a_1 + 1, \\
\rho_9 &= (a_2 + 1)(a_3 + 1), \\
\rho_{11} &= a_5 + a_4 + a_3 + a_2 + a_1 + 1.
\end{aligned}$$

At first glance, there does not appear to be any reasoning behind the choice of subscripts; note that there is no subscript for the second value. However, observe that in each case, the subscript

$n =$ (as a sum of powers of a) the number of terms with positive coefficients minus the number of terms with negative coefficients in the representation of ρ_n, when all expressions are expanded out, or if $\rho_n = \rho_n(a)$, we see that $\rho_n(1) = n$.

The reason ρ does not have a subscript is that the value of n in this case would be $3 - 2 = 1$, which has been reserved for the first factor. These factors then lead to rapid calculations of values for $p(n)$. For example, since $\lambda_{10} = \rho\rho_2\rho_3\rho_7$, then

$$p(10) = 1 \cdot 2 \cdot 3 \cdot 7 = 42.$$

In the table below, we provide the content of this page.

$$\begin{aligned}
p(1) &= 1, & \lambda_1 &= \rho_1, \\
p(2) &= 2, & \lambda_2 &= \rho_2, \\
p(3) &= 3, & \lambda_3 &= \rho_3, \\
p(4) &= 5, & \lambda_4 &= \rho_5, \\
p(5) &= 7, & \lambda_5 &= \rho_7\rho, \\
p(6) &= 11, & \lambda_6 &= \rho_1\rho_{11}, \\
p(7) &= 15, & \lambda_7 &= \rho_3\rho_5, \\
p(8) &= 22, & \lambda_8 &= \rho_1\rho_2\rho_{11}, \\
p(9) &= 30, & \lambda_9 &= \rho_2\rho_3\rho_5, \\
p(10) &= 42, & \lambda_{10} &= \rho\rho_2\rho_3\rho_7, \\
p(11) &= 56, & \lambda_{11} &= \rho_4\rho_7(a_5 - a_4 + a_2),
\end{aligned}$$

$$
\begin{aligned}
p(12) &= 77, & \lambda_{12} &= \rho_7\rho_{11}(a_4 - 2a_3 + 2a_2 - a_1 + 1),\\
p(13) &= 101, & \lambda_{13} &= \rho\rho_1\,(a_{10} + 2a_9 + 2a_8 + 2a_7 + 3a_6\\
&&&\qquad +4a_5 + 6a_4 + 8a_3 + 9a_2 + 9a_1 + 9)\,,\\
p(14) &= 135, & \lambda_{14} &= \rho_5\rho_9(a_5 - a_3 + a_1 + 1),\\
p(15) &= 176, & \lambda_{15} &= \rho_4\rho_{11}(a_7 - a_6 + a_4 + a_1),\\
p(16) &= 231, & \lambda_{16} &= \rho_3\rho_7\rho_{11}(a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3),\\
p(17) &= 297, & \lambda_{17} &= \rho_9\rho_{11}(a_7 - a_6 + a_3 + a_1 - 1),\\
p(18) &= 385, & \lambda_{18} &= \rho_5\rho_7\rho_{11}(a_6 - 2a_5 + a_4 + a_3 - a_2 + 1),\\
p(19) &= 490, & \lambda_{19} &= \rho_1\rho_2\rho_5\rho_7(a_9 - a_7 + a_4 + 2a_3 + a_2 - 1),\\
p(20) &= 627, & \lambda_{20} &= \rho\rho_3\rho_{11}(a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3),\\
p(21) &= 792, & \lambda_{21} &= \rho\rho_3\rho_4\rho_{11}(a_8 - a_6 + a_4 + a_1 + 2).
\end{aligned}
$$

8. Further Entries on Page 59

Further down page 59, Ramanujan offers the quotient (with one misprint corrected)

$$
\begin{aligned}
\Big(1 &+ q(a_1-2) + q^2(a_2-a_1) + q^3(a_3-a_2) + q^4(a_4-a_3) + \cdots\\
&- \big(q^3(a_1-2) + q^5(a_2-a_1) + q^7(a_3-a_2) + q^9(a_4-a_3) + \cdots\big)\\
&+ \big(q^6(a_1-2) + q^9(a_2-a_1) + q^{12}(a_3-a_2) + q^{15}(a_4-a_3) + \cdots\big)\\
&- \big(q^{10}(a_1-2) + q^{14}(a_2-a_1) + q^{18}(a_3-a_2) + q^{22}(a_4-a_3) + \cdots\big)\\
&+ \big(q^{15}(a_1-2) + q^{20}(a_2-a_1) + q^{25}(a_3-a_2) + \cdots\big)\\
&- \big(q^{21}(a_1-2) + \cdots\big)\Big)\Big/\\
&(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \cdots).
\end{aligned}
\tag{8.1}
$$

In more succinct notation, (8.1) can be rewritten as

$$
\frac{1 - \sum\limits_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(a_{n+1} - a_n)}{(q;q)_\infty}, \tag{8.2}
$$

where now $a_0 := 2$. Scribbled underneath (8.1) are the first few terms of (5.1) through q^5. Thus, although not claimed by Ramanujan, (8.1) is, in fact, equal to $F_a(q)$. We state this in the next theorem, with a_n replaced by A_n.

Theorem 8.1. *If A_n is given by (3.2), then, if $|q| < \min(|a|, 1/|a|)$,*

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = 1 - \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(A_{n+1}-A_n). \tag{8.3}$$

It is easily seen that Ramanujan's Theorem 8.1, which we prove in the next section, is equivalent to a theorem discovered independently by R. J. Evans [10, eq. (3.1)], V. G. Kač and D. H. Peterson [17, eq. (5.26)], and Kač and M. Wakimoto [18, middle of p. 438]. As remarked in [17], the identity, in fact, appears in the classic text of J. Tannery and J. Molk [21, Sect. 486].

Theorem 8.2. *Let*

$$r_k = (-1)^k q^{k(k+1)/2}. \tag{8.4}$$

Then

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{r_k(1-a)}{1-aq^k}. \tag{8.5}$$

A notable feature of the authors' [6] second method, based on Theorem 8.1 or Theorem 8.2, for establishing Ramanujan's five congruences is that elegant identities arise in the proofs. For example, in the proof of Theorem 2.1, we need to prove that

$$\sum_{k=-\infty}^{\infty} r_k \frac{q^k-1}{1+q^{4k}} = q\frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^2,-q^{14})$$

and

$$\sum_{k=-\infty}^{\infty} r_k \frac{q^k+1}{1+q^{4k}} = \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^6,-q^{10}),$$

where r_k is defined by (8.4). To prove Theorem 2.2, we need to prove

$$\sum_{k=-\infty}^{\infty} r_k \frac{q^k-1}{1+q^{3k}+q^{6k}} = q(q;q)_\infty \frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}$$

and two similar identities.

On page 59, below the list of factors and above the two foregoing series, Ramanujan records two further series, namely,

$$S_1(a,q) := \frac{1}{1+a} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^n q^{n(n+1)/2}}{a+q^n} \right) \tag{8.6}$$

and

$$S_2(a,q) := 1 + \sum_{m=1,n=0}^{\infty} (-1)^{m+n} q^{m(m+1)/2+mn}(a_{n+1}+a_n), \qquad (8.7)$$

where here $a_0 := 1$. No theorem is claimed by Ramanujan, but the following theorem, to be proved in the next section, holds.

Theorem 8.3. *With* $S_1(a,q)$ *and* $S_2(a,q)$ *defined by* (8.6) *and* (8.7), *respectively,*

$$(1+a)S_1(a,q) = S_2(a,q) = F_{-a}(q).$$

9. Proofs of Theorems 8.1 and 2.1

Proof of Theorem 8.1. Our proof is different from that of Evans [10], Kač and Peterson [17], and Kač and Wakimoto [18]. We employ the partial fraction decomposition

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}(1+q^n) \times \left\{1 - \frac{1-q^n}{1+q^n}\sum_{m=0}^{\infty} a^m q^{mn} - \frac{1-q^n}{1+q^n}\sum_{m=1}^{\infty} a^{-m}q^{mn}\right\}, \qquad (9.1)$$

found in Garvan's paper [13, eq. (7.16)]. From (9.1), we find that

$$\begin{aligned}
&\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\Bigg((1+q^n) \\
&\qquad - (1-q^n)\sum_{m=0}^{\infty} a^m q^{mn} - (1-q^n)\sum_{m=1}^{\infty} a^{-m}q^{mn}\Bigg) \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(2 - (1-q^n)\sum_{m=0}^{\infty} q^{mn}(a^m + a^{-m})\right) \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(2 - \sum_{m=0}^{\infty} q^{mn}A_m + \sum_{m=0}^{\infty} q^{(m+1)n}A_m\right) \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(-\sum_{m=1}^{\infty} q^{mn}A_m + \sum_{m=1}^{\infty} q^{mn}A_{m-1}\right) \\
&= 1 + \sum_{m,n=1}^{\infty}(-1)^n q^{n(n-1)/2+mn}\left(A_{m-1} - A_m\right)
\end{aligned}$$

$$= 1 - \sum_{m=0,n=1}^{\infty} (-1)^n q^{n(n+1)/2+mn} \left(A_{m+1} - A_m\right),$$

which is (8.3), but with the roles of m and n reversed. □

Proof of Theorem 2.1. Multiply (8.6) throughout by $(1+a)$ to deduce that

$$\begin{aligned}
(1+a)S_1(a,q) &= 1 + (1+a)\sum_{n=1}^{\infty}\left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^n q^{n(n+1)/2}}{a+q^n}\right) \\
&= 1 + (1+a)\sum_{n=1}^{\infty}\left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^{-n} q^{n(n-1)/2}}{1+aq^{-n}}\right) \\
&= 1 + (1+a)\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} \\
&= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(1+a)}{1+aq^n} \\
&= \frac{(q;q)_\infty^2}{(-aq;q)_\infty(-q/a;q)_\infty},
\end{aligned} \tag{9.2}$$

by an application of (8.5).

Secondly,

$$\begin{aligned}
S_2(a,q) &= 1 + \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn} \\
&\qquad \times \left(-(-a)^{n+1} - (-a)^{-n-1} + (-a)^n + (-a)^{-n}\right) \\
&= \frac{(q;q)_\infty^2}{(-aq;q)_\infty(-q/a;q)_\infty},
\end{aligned} \tag{9.3}$$

by Theorem 8.1. Thus, (9.2) and (9.3) yield Theorem 2.1. □

10. Conclusion

From the abundance of material in the lost notebook on factors of the coefficients λ_n of the generating function (2.1) for cranks, $F_a(q)$, Ramanujan clearly was eager to find some general theorems with the likely intention of applying them in the special case of $a = 1$ to determine arithmetical properties of the partition function $p(n)$. Although he was able to derive five beautiful congruences for $F_a(q)$, the kind of arithmetical theorem that he was seeking evidently eluded him. Indeed, general theorems on the divisibility of λ_n by sums of powers of a appear extremely difficult, if not impossible, to obtain. Moreover, demonstrating that the tables in Section 5.6 are complete seems to be a formidable challenge.

Garvan discovered a 5-dissection of $F_a(q)$, where a is any primitive 10th root of unity, in [14, eq. (2.16)]. This is, to date, the only dissection identity for the generating function of cranks that does not appear in Ramanujan's lost notebook. It would also be interesting to uncover new dissection identities of $F_a(q)$ when a is a primitive root of unity of order greater than 11.

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